Interval numbers of powers of block graphs

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Abstract

The interval number of a graph $G$ is the minimum $t$ such that each vertex of $G$ can be assigned a set that is the union of at most $t$ intervals on the real line so that distinct vertices are adjacent if and only if their corresponding sets intersect. A graph with interval number one is an interval graph. We prove that the interval number of the $k$th power of a block graph is at most $k + 1$. We also characterize block graphs whose $k$th powers are interval graphs. Since trees are block graphs and are their own first powers, these results generalize those of Trotter and Harary that the interval number of a tree is at most two, and a tree is an interval graph if and only if it is a caterpillar.

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1. Introduction

The intersection graph of a family $\mathcal{F}$ of sets is the graph with a vertex for each set in $\mathcal{F}$ such that distinct vertices are adjacent if and only if their corresponding sets intersect. Interval graphs are the intersection graphs of families of intervals on a line. They have applications in many fields, such as biology and computer science.

To generalize the concept of interval graphs, Trotter and Harary [29] introduced interval numbers of graphs. The interval number $i(G)$ of a graph $G$ is the least integer $t$ such that $G$ is the intersection graph of a family $\mathcal{F}$ in which each set is the union of...
at most \( t \) intervals. Such a family \( \mathcal{F} \) is a \textit{multi-interval representation} or a \textit{\( t \)-interval representation} of \( G \). The interval graphs are the graphs with interval number 1. Previous results on interval numbers include upper bounds \([1–9,11,12,18,21–28,30]\), lower bounds \([6,7,18,19]\), exact values \([4,6,13,29]\), and NP-completeness \([31]\).

Trotter and Harary \([29]\) proved that the interval number of a tree is at most 2 and that a tree is an interval graph if and only if it is a caterpillar (a tree whose edges are incident to a single path). Our paper generalizes these results to powers of block graphs. A block of a graph is a maximal subgraph that has no cut-vertices. A graph is a \textit{block graph} if it is the intersection graph of the family of blocks of some graph. The \textit{kth power} of a graph \( G \) is the graph \( G^k \) whose vertex set is \( V(G) \) and whose edge set is \( \{xy: 1 \leq d_G(x,y) \leq k\} \), where \( d_G(x,y) \) denotes the distance between \( x \) and \( y \), meaning the least length (number of edges) of an \( x,y \)-path.

Harary \([11]\) proved that a graph is a block graph if and only if all its blocks are complete graphs; thus trees are block graphs. We prove that the interval number of the \( k \)th power of a block graph is at most \( k + 1 \). We actually establish a slightly stronger result that the \( k \)th power of a block graph is the union of \( k + 1 \) interval graphs. The minimum number of interval graphs whose union is \( G \) is called the \textit{track number} of \( G \) in \([10]\) (see \([15,16]\) for more results); it is an upper bound on \( i(G) \). For the sharpness of this upper bound, we present a tree \( T \) with \( i(T^2) = 3 \). It remains unknown whether there exists any power of any block graph whose interval number exceeds 3. We also characterize block graphs whose \( k \)th powers are interval graphs. Our results reduce to those of \([29]\) when \( k = 1 \) and the given block graph is a tree.

An \textit{interval ordering} of a graph \( G \) is an ordering \( v_1, \ldots, v_n \) of its vertices such that

\[
i < \ell < j \quad \text{and} \quad v_i v_j \in E(G) \implies v_\ell v_j \in E(G).
\]

It is easy to show that a graph is an interval graph if and only if it has an interval ordering. This was observed by Ramalingam and Pandu Rangan \([22]\) (studying domination), by Jacobson et al. \([14]\) (studying tolerance intersection graphs), and by Olariu \([20]\). Many papers study algorithms or theorems for interval graphs by using interval representations directly, often working with the endpoints of the intervals. For us, it is more convenient to use interval orderings.

We sometimes use a more difficult characterization due to Lekkerkerker and Boland \([17]\): a graph is an interval graph if and only it is a chordal graph with no asteroidal triple. A \textit{chordal graph} is a graph with no “chordless cycle”, meaning a cycle of length at least 4 as an induced subgraph. An \textit{asteroidal triple} is a set of three distinct vertices such that each pair lies on some path containing no neighbor of the third.

2. Upper bounds for powers of block graphs

This section proves that the interval number of the \( k \)th power of a block graph \( G \) is at most \( k + 1 \). We may assume without loss of generality that \( G \) is connected. The proof then consists of two main steps.

(1) Append a path of length \( \lceil k/2 \rceil \) to get a supergraph \( B \) that is also a block graph. Prove that \( i(G^k) \leq i(B^k) \) so that proving \( i(B^k) \leq k + 1 \) will suffice.
(2) Prove that $B^k$ is the union of $k + 1$ interval graphs $H_0, H_1, \ldots, H_k$.

We first discuss step (1). In general, the $k$th power of an induced subgraph $H$ of $G$
 need not be an induced subgraph of $G^k$, and $i(H^k) \leq i(G^k)$ may not hold. Consider
the 6-cycle $C_6$ and the graph $W$ formed by adding a new vertex adjacent to all of $V(C_6)$. The graph $C_6^2$
 has a chordless 4-cycle and hence is not an interval graph, but $W^2$ is a complete graph and hence is an interval graph.

For some kind of graphs $G$, we do have $i(H^k) \leq i(G^k)$ when $H$ is an induced
subgraph of $G$. A graph $G$ is distance-hereditary if $d_C(x, y) = d_G(x, y)$ for every two
vertices $x$ and $y$ in each connected induced subgraph $C$ of $G$.

**Proposition 1.** If $H$ is an induced subgraph of a distance-hereditary graph $G$ (such as
when $G$ is a block graph), then $i(H^k) \leq i(G^k)$ for every positive integer $k$.

**Proof.** If $x$ and $y$ are vertices in a component $C$ of $H$, then $d_C(x, y) = d_G(x, y)$. Hence $xy \in E(C^k)$ if and only if $xy \in E(G^k)$. Thus $C^k$ is an induced subgraph of $G^k$ and $i(C^k) \leq i(G^k)$. Consequently, $i(H^k) \leq i(G^k)$.

Since every block of a block graph $G$ is a complete subgraph, vertices $x$ and $y$ in the
same component are joined by a unique shortest path whose internal vertices are all cut-vertices. Hence $x$ and $y$ lie in the same component of an induced subgraph $H$
if and only if all those cut-vertices also lie in $H$, which yields $d_H(x, y) = d_G(x, y)$. \(\square\)

To discuss distances in block graphs, we introduce an auxiliary graph. The block-
vertex graph of a graph $G$ is the graph $G^*$ whose vertex set is $V(G) \cup \{A: A$ is a block
of $G\}$ and whose edge set is $\{xA: A$ is a block and $x$ is a vertex in $A\}$. The vertices of
degree 1 in the block-vertex graph correspond to the vertices in $G$ belonging to only
one block, and deleting them yields the more common “block-cutpoint graph” of $G$.

The block-vertex graph is bipartite, since edges join blocks to vertices. It is also
acyclic, since the union of the blocks on a cycle in $G^*$ would form a larger subgraph
of $G$ with no cut-vertex. Hence the block-vertex graph of a connected graph is a tree.

We have observed that any two vertices $u$ and $v$ in a connected block graph $B$
are joined by a unique shortest path $P$ and that the internal vertices on $P$ are cut-vertices.
Furthermore, successive vertices on $P$ lie in a common block of $B$, and these blocks are
distinct. Including these blocks yields the unique $u, v$-path in $B^*$. Therefore, $d_{B^*}(u, v) = 2d_B(u, v)$.

Let $T$ be a tree rooted at $r$. For any vertex $v$ in $T$, the vertices on the unique $r, v$-path
in $T$ (including $v$) are the ancestors of $v$; and $v$ is a descendant of its ancestors. The
common ancestors of any two vertices $u$ and $v$ form a path $r, \ldots, w$, and $w$ is their least
common ancestor. The ancestor adjacent to a vertex is its parent, and the descendants
adjacent to a vertex are its children.

We are now ready to consider the main result of this section.

**Theorem 2.** If $G$ is a block graph and $k$ is a positive integer, then $i(G^k) \leq k + 1$.

**Proof.** We may assume that $G$ is connected. Let $B$ be the block graph obtained from
$G$ by identifying a vertex of $G$ with one end of a path $P$ of length $\lceil k/2 \rceil$. Let $r$
denote the other end of \( P \). Since \( G \) is an induced subgraph of \( B \), Proposition 1 yields \( i(G^k) \leq i(B^k) \), and it suffices to prove that \( i(B^k) \leq k + 1 \). Our plan is to express \( B^k \) as the union of \( k + 1 \) subgraphs that are interval graphs. We begin by associating a subgraph \( G_x \) of \( B^k \) with each vertex \( x \) of \( B \) and showing that these graphs are interval graphs.

View the block-vertex graph \( B^* \) of \( B \) as a tree \( T \) rooted at \( r \). (See Fig. 1 for an example of \( B \) and \( T \) and the analysis of such a graph \( G_x \) as presented next.)

Recall that \( d_T(u,v) = 2d_B(u,v) \) whenever \( u,v \in V(B) \). Let \( D(u) \) denote the set of descendants of \( u \) in \( T \); this generally contains vertices of \( B \) and blocks of \( B \). For \( x \in V(B) \), let \( V_x = D(x) \cap N_B(x) \), where \( N_B(x) \) denotes the set of neighbors of \( x \) in \( G \). Thus \( V_x \) consists of the descendants of \( x \) in \( T \) that are vertices of \( B \) within distance \( k \) of \( x \) in \( B \); these are marked with squares in Fig. 1.

For \( v \in V_x \), let \( h_x(v) \) be the vertex on the \( x,v \)-path in \( T \) that is halfway from \( x \) to \( v \); this exists since \( d_T(x,v) \) is even. Let \( G_x \) be the graph whose vertex set is \( V_x \) and whose edge set \( E_x \) is \{uv: one of \( h_x(u),h_x(v) \) is a descendant of the other in \( T \)\}. If \( h_x(u) \) is a descendant of \( h_x(v) \), for example, then \( d_T(v,h_x(v)) = d_T(h_x(v),x) \) yields \( d_T(u,v) \leq d_T(u,x) \leq 2k \). Hence \( G_x \subseteq B^k \).

We claim that \( G_x \) is an interval graph. Let \( T_x \) be the subtree of \( T \) induced by \( \{h_x(v): v \in V_x \} \). Perform a postorder traversal of \( T_x \); this visits the vertices in the subtree with a given root by recursively visiting the vertices in the subtrees rooted at
its children from left to right and then visiting the root last. This yields an ordering \( \sigma \) of \( V(T_x) \) such that (PT1) and (PT2) below hold:

(PT1) If \( u \in D(v) - \{v\} \), then \( \sigma(u) < \sigma(v) \).

(PT2) If \( \sigma(u) < \sigma(v) < \sigma(w) \) and \( u \in D(w) \), then \( v \in D(w) \).

For \( y \in V(T_x) \), let \( I_y = \{v \in V_x : h_x(v) = y\} \); these sets partition \( V_x \). Let \( \sigma' \) be the ordering of \( V_x \) obtained from \( \sigma \) by replacing each vertex \( y \) in \( \sigma \) with the vertices of \( I_y \) in any order. Show that \( \sigma' \) is an interval ordering of \( G_x \) will prove that \( G_x \) is an interval graph.

Consider \( v_1, v_2, v_3 \in V_x \) such that \( \sigma'(v_1) < \sigma'(v_2) < \sigma'(v_3) \) and \( v_1, v_3 \in E_x \). The construction of \( \sigma' \) yields \( \sigma(h_x(v_1)) < \sigma(h_x(v_2)) < \sigma(h_x(v_3)) \). Since \( v_1, v_3 \in E_x \), the definition of \( E_x \) and (PT1) imply that \( h_x(v_1) \in D(h_x(v_3)) \). Now (PT2) yields \( h_x(v_2) \in D(h_x(v_3)) \), and hence \( v_2, v_3 \in E_x \). Thus \( \sigma' \) is an interval ordering of \( G_x \), and \( G_x \) is an interval graph.

Let \( \mathcal{G} = \{G_x : x \in V(B)\} \). We use these graphs to express \( B^k \) as the union of \( k + 1 \) interval graphs. If \( d_T(r, x) \equiv d_T(r, y) \pmod{2k + 2} \), then \( V_x \cap V_y = \emptyset \), since \( V_x \) contains only descendants of \( x \) in \( T \) within distance \( 2k \) of \( x \) in \( T \). Therefore, with \( H_j = \{G_x : d_T(r, x) \equiv 2j \pmod{2k + 2}\} \), we have that each \( H_j \) is an interval graph for \( j = 0, 1, \ldots, k \). Since we have observed that \( G_x \subseteq B^k \) for all \( x \in V(B) \), it remains only to show that \( B^k \subseteq \bigcup_{j=0}^{k} H_j \).

Given an edge \( uv \in E(B^k) \), let \( y \) be the least common ancestor of \( u \) and \( v \) in \( T \). Without loss of generality, we may assume that \( d_T(y, v) \leq d_T(y, u) \). We claim that \( d_T(y, v) \leq d_T(y, r) \). If \( v \in V(P) \), then \( y = v \) and the claim holds. Otherwise, \( y \) is a descendant of all the vertices in \( P \), and \( d_T(y, r) \geq k \). This suffices, since the computation \( 2d_T(y, v) \leq d_T(y, v) + d_T(y, u) = d_T(u, v) = 2d_B(u, v) \leq 2k \) yields \( d_T(y, v) \leq k \).

With \( d_T(y, r) \geq d_T(y, v) \), we may let \( x \) be the ancestor of \( y \) in \( T \) such that \( d_T(x, y) = d_T(x, v) \). Since \( d_T(x, v) \) is even and \( v \in V(B) \), also \( x \in V(B) \). Since \( d_T(x, v) \leq d_T(x, u) = d_T(u, v) = 2d_B(u, v) \leq 2k \), we have \( u, v \in V_x \). Also, \( h_x(u) \in D(y) = D(h_x(v)) \). Thus \( uv \in E_x \). This yields \( B^k \subseteq \bigcup H_j \), completing the proof. \( \square \)

Notice that the proof of Theorem 2, in fact, gives a slightly stronger result that the \( k \)th power of a block graph is the union of \( k + 1 \) interval graphs.

The following theorem shows that Theorem 2 is sharp for \( k = 2 \). We have not been able to show that the theorem is sharp for larger \( k \). Indeed, it remains unknown whether there exists any power of any block graph whose interval number exceeds \( 3 \). Some of the arguments in the proof hold for more general \( k \).

**Theorem 3.** \( i(T^2) = 3 \), where \( T \) is the rooted tree in which the root \( r \) has distance 4 from the leaves and every non-leaf vertex has nine children.

**Proof.** Let \( L_t \) be the set of 9' vertices at distance \( t \) from \( r \) in \( T \). Suppose that \( i(T^2) \leq 2 \), and let \( f \) be a 2-interval representation of \( T^2 \), with \( f(x) \) being the set of points in the intervals assigned to \( x \). For \( S \subseteq V(T^2) \), let \( f(S) = \bigcup_{v \in S} f(v) \). Let \( D(x) \) denote the set of descendants of \( x \), as before.
We observe first that
\[ \text{If } x \in L_1 \cup L_2, \text{ then } f(x) \not\subseteq \bigcup_{y \in V(T) \setminus D(x)} f(y). \quad (*) \]
If the containment holds, then choose \( z \in D(x) \cap L_{d(x,r)=2} \). Now \( d_T(z,x) = 2 \) requires \( f(z) \cap f(x) \neq \emptyset \), but \( d(z,y) > 2 \) requires \( f(z) \cap f(y) = \emptyset \) for all \( y \in V(T) \setminus D(x) \).

Next we define a special set \( S \subseteq L_2 \). Note that \( f(v) \cap f(r) \neq \emptyset \) when \( v \in L_2 \). For each maximal interval \( I \) in \( f(r) \), choose a vertex \( v \in L_2 \) such that \( f(v) \) contains the leftmost point of \( I \setminus f(L_2) \) and a vertex \( v' \in L_2 \) such that \( f(v') \) contains the rightmost point of \( I \setminus f(L_2) \) (if \( I \setminus f(L_2) \neq \emptyset \)). Let \( S \) be the set of all vertices so chosen; since \( f(r) \) is the union at most two intervals, we have \( |S| \leq 4 \).

Let \( L^*_1 \) be the set of vertices in \( L_1 \) whose least common ancestor with all vertices of \( S \) is \( r \). For \( u \in L^*_1 \) with \( 1 \leq t \leq 2 \), we have \( d_T(u,S) > 2 \), and hence every maximal interval in \( f(u) \) that intersects \( f(r) \) is entirely contained in \( f(r) \).

We next prove that for \( u \in L^*_1 \), some portion of \( f(u) \) contained in \( f(r) \) intersects \( f(D(u) \cap L_2) \). Otherwise, the children of \( u \) all have intervals outside \( f(r) \) that intersect \( f(u) \) in addition to their intervals contained entirely in \( f(r) \) that do not intersect \( f(u) \). Since \( k=2 \), there is only one interval \( I \) for \( u \) outside \( f(r) \). Let \( w \) and \( z \) be the children of \( u \) whose intervals intersecting \( I \) have leftmost and rightmost endpoints, respectively. Now \( u \) has a third child \( x \), and we have shown that \( f(x) \subseteq f(r) \cup f(u) \cup f(w) \cup f(z) \). This contradicts \((*)\) for \( x \in L_2 \).

For each \( u \in L^*_1 \), we now have one child \( u' \in L^*_2 \) for which there is an interval intersecting \( f(u) \) inside \( f(r) \). These intervals generated by distinct \( u \in L^*_1 \) are pairwise disjoint, since children of distinct elements of \( L^*_1 \) have distance 4 in \( T \). Since \( |S| \leq 4 \), we have \( |L^*_1| \geq 5 \). We index these vertices as \( u_1, \ldots, u_5 \) in the left-to-right order of the intervals for \( u_1', \ldots, u_5' \) in \( f(r) \). Since \( u_t \) and \( u'_t \) are non-adjacent in \( T^2 \) when \( t \neq s \), within \( f(r) \) we must end \( f(u_1) \) before \( f(u'_2) \) starts, end \( f(u'_2) \) before \( f(u_3) \) starts, etc. Hence \( u_1, u_3, u_5 \) are assigned pairwise disjoint intervals within \( f(r) \).

On the other hand, \( u_1, u_3, u_5 \) are pairwise adjacent in \( T^2 \). Therefore, their assigned intervals outside \( f(r) \) are pairwise intersecting. Let \( w \) and \( z \) be the vertices among \( \{u_1, u_3, u_5\} \) whose intervals among these three are assigned the leftmost and rightmost points (possibly \( w=z \)), and choose \( x \in \{u_1, u_3, u_5\} - \{w,z\} \). Now \( f(x) \subseteq f(w) \cup f(z) \cup f(r) \), which contradicts \((*)\) for \( x \in L_1 \). \( \square \)

Note that we did not need the full tree in Theorem 3. It suffices for the root to have nine children, each vertex of \( L_1 \) to have three children, and each vertex of \( L_2 \cup L_3 \) to have one child.

3. Powers of block graphs that are interval graphs

This section characterizes block graphs whose \( k \)th powers are interval graphs.

Given a graph \( G \), let \( G' \) denote the graph obtained from \( G \) by deleting all simplicial vertices, which are the vertices whose neighbors form a clique. If \( G \) is a block graph, then \( G' \) is the graph obtained by deleting all the non-cut vertices, and \( G' \) is a block
graph. Define $G^{(n)}$ recursively by letting $G^{(0)}$ be $G$ and letting $G^{(n)} = (G^{(n-1)})'$ for $n \geq 1$. Note that $G'$ is analogous to Harary’s notion of “derived graph”, which is the graph obtained by deleting all leaves. Our notation thus evokes the notation for iterative derivations of functions.

A clique-path is a connected block graph in which every cut-vertex is in exactly two blocks and every block contains at most two cut-vertices. A path is simply a clique-path in which every block has two vertices.

**Theorem 4.** If $B$ is a connected block graph and $k$ is a positive odd (even) integer, then $B^k$ is an interval graph if and only if $B^\lceil k/2 \rceil$ is a path (clique-path).

**Proof.** Let $m = \lceil k/2 \rceil$.

*Necessity:* We prove the contrapositive. Let $k$ be odd (even), and suppose that $B^m$ is not a path (clique-path). We prove that $B^k$ is not an interval graph. For odd $k$, since $B^m$ is not a path, it has an induced subgraph isomorphic to $G_1$ or $G_2$ in Fig. 2. For even $k$, since $B^m$ is not a clique-path, it has an induced subgraph isomorphic to $G_2$ or $G_3$ in Fig. 2.

Every vertex of $B^m$ is non-simplicial in $B^{m-1}$, since it appears in $B^m$. Hence $x_m$ is a cut-vertex in $B^{m-1}$, and there exists $x_{m-1} \in V(B^{m-1})$ whose only neighbor in $V(B^m)$ is $x_m$. Similarly, for $y_m$ and $z_m$ we obtain $y_{m-1}$ and $z_{m-1}$. Furthermore, $\{x_{m-1}, y_{m-1}, z_{m-1} \}$ is independent. Repeating the argument $m-1$ more times yields in $B$ an induced subgraph isomorphic to $H_1$ or $H_2$ ($H_2$ or $H_3$) in Fig. 3 when $k$ is odd (even). The resulting set $\{x_0, y_0, z_0 \}$ is an asteroidal triple in $B$, and hence $B$ is not an interval graph, by the Lekkerkerker–Boland characterization [17].
Su7SOciency: Given that \( B^k \) is a path (clique-path) and \( k \) is odd (even), we produce an interval ordering for \( B^k \). Let \( T \) be the block-vertex graph \( B^* \) of \( B \); recall that \( 2d_B(u, v) = d_T(u, v) \) for \( u, v \in V(B) \). Hence \( B^k \) is an induced subgraph of \( T^{2k} \).

Deleting the simplicial (non-cut) vertices from a block graph eliminates peripheral blocks (those with at most one cut-vertex). Thus \((B')^* = (B^*)^{(2)}\). Induction now yields \((B^m)^* = T(2m)\).

The block-vertex graph of a clique-path is a caterpillar, and the derived graph of a caterpillar is a path. Hence whether \( k \) is even or odd the tree \( T(k + 1) \) is a path. Let \( X = V(T(k + 1)) \), consisting of vertices \( x_0, \ldots, x_t \) in order along the path.

View \( T \) as a tree rooted at \( x_0 \). We construct an ordering \( \sigma \) of \( V(T) \) that contains an interval ordering of \( B^k \). Perform a breadth-first search (BFS) of \( T \), starting at \( x_0 \), extending the ordering at each iteration by adding the children of the oldest unexplored reached node, and listing each \( x_i \in V(P) \) as the last vertex reached in level \( i \). Let \( \sigma \) denote the reverse of this ordering, listing \( V(T) \) as \( v_1, \ldots, v_n \).

For each \( v \in V(T) \), let \( s(v) \) be the index \( j \) such that the path from \( x_0 \) to \( v \) in \( T \) leaves \( X \) at \( x_j \). By its construction, the ordering \( \sigma \) satisfies the following two conditions, where RBFS stands for “Reverse BFS”.

(RBFS1) If \( d_T(x_0, v_i) > d_T(x_0, v_j) \) or \( d_T(x_0, v_i) = d_T(x_0, v_j) \) with \( s(v_i) > s(v_j) \), then \( i < j \).

(RBFS2) If \( i < j < p \) and \( d_T(x_0, v_i) = d_T(x_0, v_j) = d_T(x_0, v_p) \) and \( s(v_i) = s(v_j) = s(v_p) \), then \( v_i \) is a descendant of the least common ancestor of \( v_i \) and \( v_p \).

We claim that the order in which vertices of \( B \) appear in \( \sigma \) is an interval ordering of \( B^k \). Suppose that \( i < j < p \) and \( v_i v_p \in E(B^k) \). Let \( y \) be the least common ancestor of \( v_i \) and \( v_p \). We prove in each case that \( v_i v_p \in E(B^k) \).

Case 1: \( y \) is an ancestor of \( v_j \) (see Fig. 4(a)). Since \( i < j \), (RBFS1) implies that \( d_T(x_0, v_j) \leq d_T(x_0, v_i) \). Now \( d_T(v_j, y) \leq d_T(v_i, y) \), and hence \( d_T(v_j, v_p) \leq d_T(v_j, y) + d_T(v_i, v_p) \leq d_T(v_i, v_p) = 2d_B(v_i, v_p) \leq 2k \), as desired.

Case 2: \( y \) is not an ancestor of \( v_j \) (see Fig. 4(b)). Let \( z \) be the least common ancestor of \( y \) and \( v_j \); it is a proper ancestor of \( y \). Let \( z_1 \) and \( z_2 \) be the children of \( z \) that are ancestors of \( y \) and \( v_j \), respectively. Since \( i < j < p \), we have \( d_T(x_0, v_p) \leq d_T(x_0, v_j) \leq d_T(x_0, v_i) \), and hence \( d_T(v_p, z) \leq d_T(v_j, z) \leq d_T(v_i, z) \). To
obtain \( d_B(v_j, v_p) \leq k \) and \( v_jv_p \in E(B^k) \), it suffices to show that \( d_T(v_j, z) \leq k \) or \( d_T(v_p, z) < d_T(v_j, z) \leq k + 1 \), since \( d_T(v_j, v_p) = 2d_B(v_j, v_p) \). By the construction of \( X \), for each \( v \in V(T) \) we have \( d_T(v, x_i(v)) \leq k + 1 \).

**Subcase 2.1:** \( X \) does not contain \( z \). By the construction of \( X \), \( d_T(v_j, z) \leq k \).

**Subcase 2.2:** \( X \) contains \( z \) and \( z_1 \) but not \( z_2 \). Here \( s(v_j) < s(v_p) \) and \( d_T(v_j, z) \leq k + 1 \). Since \( j < p \), (RBFS1) implies that \( d_T(v_p, z) < d_T(v_j, z) \).

**Subcase 2.3:** \( X \) contains \( z \) and \( z_2 \) but not \( z_1 \). Here \( s(v_j) < s(v_j) \) and \( d_T(v_j, z) \leq k + 1 \). Since \( i < j \), (RBFS1) implies that \( d_T(v_i, z) < d_T(v_i, z) \).

**Subcase 2.4:** \( X \) contains \( z \) but neither \( z_1 \) nor \( z_2 \). Here \( s(v_j) = s(v_j) = s(v_j) \) and \( d_T(v_j, z) \leq d_T(v_j, z) \leq k + 1 \). Since \( v_j \) is not an ancestor of the least common ancestor of \( v_i \) and \( v_p \), (RBFS2) implies that \( d_T(v_p, z) < d_T(v_j, z) \) or \( d_T(v_j, z) < d_T(v_j, z) \). □

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**References**


